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Prescribing singularities of maximal surfaces via a singular Björling representation formula

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Abstract

We derive a proper formulation of the singular Björling problem for spacelike maximal surfaces with singularities in the Lorentz–Minkowski 3-space which roughly asks whether there exists a maximal surface that contains a prescribed curve as singularities, and then provide a representation formula which solves the problem in an affirmative way. As consequences, we construct many kinds of singularities of maximal surfaces and deduce some properties of the maximal surfaces related to the singularities due to the geometry of the Gauss map.

MSC: 53A35

Keywords: Maximal surfaces; Singularities; Björling formula

1. Introduction

Spacelike maximal surfaces in Lorentz–Minkowski 3-space $\mathbb{L}^3$ are similar to minimal surfaces in Euclidean 3-space $\mathbb{E}^3$ in that they also admit Weierstrass representation formula [14]. But they are very different from the minimal surfaces in that they have naturally arising singularities due to the geometry of the Gauss map of spacelike surfaces in $\mathbb{L}^3$. The presence of singularities seems to be the cause of neglect for maximal surfaces until recent years, but the singularities are main ingredients in studying maximal surfaces. Because of the analytic continuation principle, if we can understand the local behaviors of maximal surfaces around those singularities, then in principle we can understand their global behaviors. Therefore, it is valuable to have a tool for constructing maximal surfaces which contains a prescribed set in $\mathbb{L}^3$ as singularities.

Study of singularities of spacelike maximal surfaces has gained momentum recently. Fujimori, Saji, Umehara and Yamada showed that the generic singularities of the *maxfaces* are cuspidal edges, swallowtails, and cuspidal crosscaps [9,17]. Fernández, López, and Souam studied isolated singularities [5–8].

The purpose of this article is to investigate the singularities of maximal surfaces in $\mathbb{L}^3$ from the viewpoint of the singular Björling problem which asks, roughly speaking, whether there exists a spacelike maximal surface that

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Fig. 1. Maximal surfaces which contain prescribed null curves as singularities.

contains a prescribed curve in $\mathbb{L}^3$ as a singular set and prescribed null directions on the curve as the normal directions to the surface. We derive a proper formulation of this problem in Section 3 and answer it affirmatively in Theorem 3.2 (see Fig. 1).

Very recently, the Björling problem has been studied for various kinds of surfaces which admit a holomorphic representation formula [1–3,10,11,15,16,18]. In particular, Alías, Chaves, and Mira studied the (regular) Björling problem for maximal surfaces [2], and Gálvez and Mira studied the Björling problem for embedded isolated singularities of flat surfaces in hyperbolic 3-space [11].

A direct consequence of the singular Björling representation formula is that we now have a general method of constructing singularities of maximal surfaces, which is a very important tool for understanding the singularities of maximal surfaces.

Fernández, López, and Souam showed that maximal surfaces have a reflection principle with respect to an isolated singularity. As an application of the singular Björling representation formula, we show that the reflection principle holds also with respect to a more general kind of singularity which we call a shrunken singularity. We also show that the conjugate of a shrunken singularity possesses a similar reflection property. See Theorem 4.3.

Another application of the singular Björling formula is to show that an arbitrary maximal surface with a shrunken singularity converges to a parabolic catenoid when we continuously deform the surface by rotating it around a spacelike line through the isolated singularity and applying homothety centered at the singularity with a suitable homothety factor. See Lemma 4.4.

2. Preliminaries

In this article, the order of coordinates in $\mathbb{L}^3$ is (x, y, t), where t is the time component. The metric is $\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2$.

The spacelike maximal surfaces that we construct in this article are mostly generalized maximal surfaces, so we recall a few facts from [4]:

Definition 2.1 (*[4]*). Let S be a Riemann surface and $X = (x, y, t) : S \to \mathbb{L}^3$ be nonconstant and harmonic. Suppose that at any point p of S there exists a (complex) coordinate neighborhood (U, z) such that the complexified derivatives $\phi_1 = \partial x/\partial z$, $\phi_2 = \partial y/\partial z$, $\phi_0 = \partial t/\partial z$ satisfy

$$\phi_1^2 + \phi_2^2 - \phi_0^2 = 0, \tag{2.1}$$

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_0|^2 \not\equiv 0 \quad \text{(not identically zero)}. \tag{2.2}$$

Then (X, S) is said to be a generalized maximal surface. This is the end of Definition 2.1.

A generalized maximal surface $X : \mathcal{M} \to \mathbb{L}^3$, where $\mathcal{M}$ is a Riemann surface, can be written locally as

$$\phi(z) = \left(\frac{1}{2}(g^{-1} + g), \frac{\mathrm{i}}{2}(g^{-1} - g), 1\right)\mathrm{d}h, \qquad X(z) = (x_0, y_0, t_0) + \mathrm{Re}\int_{z_0}^{z} \phi(z), \quad z \in \mathcal{U}$$

for some meromorphic function g and a holomorphic 1-form dh on some domain $\mathcal{U} \subset \mathbb{C}$ such that the poles of g with order m are zeros of dh of order at least m. The unit normal and the metric are $N = \left(\frac{\bar{g}+g}{1-g\bar{g}}, \frac{\mathrm{i}(\bar{g}-g)}{1-g\bar{g}}, \frac{1+g\bar{g}}{1-g\bar{g}}\right)$ and $ds^2 = \frac{1}{4}(|g|^{-1} - |g|)^2|dh|^2$. g is called the Gauss map of X, and (2.2) is equivalent to

$$|g| \not\equiv 1. \tag{2.3}$$

Let

$$\mathcal{A} := \{p \in \mathcal{M} : |g(p)| = 1\}, \qquad \mathcal{B} := \{p \in \mathcal{M} : dX(p) \text{ vanishes}\}$$

$p \in \mathcal{M}$ is a singular point if and only if $p \in \mathcal{A} \cup \mathcal{B}$. In general, $\mathcal{A} \cap \mathcal{B} \neq \emptyset$. Singular points in $\mathcal{B} \setminus (\mathcal{A} \cap \mathcal{B})$ are isolated but singular points in $\mathcal{A}$ cannot be isolated. This is the end of quotes from Ref. [4].

There exist many concepts in literature dealing with the singular points in $\mathcal{A}$, such as conelike singularity, lightlike singularity, isolated singularity, special singularity, to name but a few. In this article, we divide $\mathcal{A}$ into three disjoint sets as in the following definition, where $D_p(\epsilon)$ is an open disk of radius ϵ centered at p. Note that for any $p \in \mathcal{A} \setminus \{\text{critical points of } g\}$, there is some $D_p(\epsilon) \subset \mathcal{U}$ and a regular embedded curve $\gamma : I \to D_p(\epsilon) \subset \mathcal{U}$ such that $\gamma[I] = D_p(\epsilon) \cap \mathcal{A}$. See for example [4,17].

Definition 2.2. We call $p \in \mathcal{A}$ a *shrinking* singular point if there is some $D_p(\epsilon) \subset \mathcal{U}$ and a regular embedded curve $\gamma : I \to D_p(\epsilon) \subset \mathcal{U}$ such that $\gamma[I] = D_p(\epsilon) \cap \mathcal{A}$ and $X \circ \gamma$ is a single point. If p is a shrinking singular point, we call $X(p)$ a shrunken singular point, and the germ of $(X, D_p(\epsilon))$ a shrunken singularity.

We call $p \in \mathcal{A}$ a *curvilinear* singular point if there is some $D_p(\epsilon) \subset \mathcal{U}$ and a regular embedded curve $\gamma : I \to D_p(\epsilon) \subset \mathcal{U}$ such that $\gamma[I] = D_p(\epsilon) \cap \mathcal{A}$ and $X \circ \gamma$ is one-to-one on I. If p is a curvilinear singular point, we call the germ of $(X, D_p(\epsilon))$ a curvilinear singularity.

We call $p \in \mathcal{A}$ which is neither shrinking nor curvilinear an *exotic* singular point.

The concept of a shrunken singularity is closely related to that of an isolated singularity in the sense of [5–8], but the former is somewhat more general than the latter. For example, the singularities of the parabolic or hyperbolic catenoids described below are shrunken singularities but not isolated singularities. A curvilinear singular point may have vanishing dX, so it is not an admissible singular point in the sense of [17]. An example of an exotic singular point is given in Remark 3.12.

We do not require that the rank of dX is never zero, so X may not be a maxface in the sense of [17]. Since the Björling problem is a local question, we only consider generalized maximal surfaces defined on an open and simply connected domain $\mathcal{U}$ in $\mathbb{C}$ in this article. But whenever necessary and possible we apply the analytic continuation principle to extend $\mathcal{U}$ to a Riemann surface.

Rotationally invariant maximal surfaces are the simplest but still the motivating examples so we list some of their properties, a good reference for which is [2]. There are three kinds of catenoids, elliptic, parabolic, hyperbolic, depending upon their causal character of the axis of rotation.

$$\begin{aligned}
&X_{ec}(u, v) = (\cos u \sinh v, \sin u \sinh v, v), \quad (u, v) \in S^1 \times \mathbb{R},\\
&X_{pc}(a, b) = \left(b + \frac{1}{3}b^3 - a^2 b, 2ab, b - \frac{1}{3}b^3 + a^2 b\right), \quad (a, b) \in \mathbb{R}^2,\\
&X_{hc}(\alpha, \beta) = (\beta, \sin\beta \sinh\alpha, \sin\beta \cosh\alpha), \quad (\alpha, \beta) \in \mathbb{R}^2.
\end{aligned} \tag{2.4}$$

For the elliptic catenoid, the entire u-circle collapses to form a conelike singularity. For the hyperbolic catenoid, the horizontal line $\beta = n\pi$ for any $n \in \mathbb{Z}$ is mapped to a single point $(n\pi, 0, 0)$, which is not an isolated singularity but a shrunken singularity. Furthermore,

$$\partial_\alpha X_{hc}(\alpha, n\pi) = (0, 0, 0), \qquad \partial_\beta X_{hc}(\alpha, n\pi) = (1, (-1)^n \sinh\alpha, (-1)^n \cosh\alpha).$$

The collection of all the $\partial_\beta X_{hc}(\alpha, n\pi)$ as α varies from $-\infty$ to ∞ becomes a (branch of a) hyperbola in the lightcone.

The following are the Weierstrass data for the elliptic, parabolic, hyperbolic catenoids:

$$\begin{aligned}
&\phi_{ec}(z) = -\mathrm{i}(\cos z, \sin z, 1)\mathrm{d}z, \quad z = u + \mathrm{i}v,\\
&\phi_{pc}(w) = -\mathrm{i}(1 - w^2, 2w, 1 + w^2)\mathrm{d}w, \quad w = a + \mathrm{i}b,\\
&\phi_{hc}(\zeta) = -\mathrm{i}(1, \sinh\zeta, \cosh\zeta)\mathrm{d}\zeta, \quad \zeta = \alpha + \mathrm{i}\beta.
\end{aligned} \tag{2.5}$$

We notice that if z, w, ζ are real, then vectors $\mathrm{i}\phi_{ec}/\mathrm{d}z$, $\mathrm{i}\phi_{pc}/\mathrm{d}w$, $\mathrm{i}\phi_{hc}/\mathrm{d}\zeta$ become an ellipse, a parabola, or a hyperbola in the lightcone which passes through $(1, 0, 1)$ when $z = w = \zeta = 0$.

The following are the conjugates of the elliptic, parabolic, and hyperbolic catenoids, which are called the helicoid of the first kind, Cayley's ruled surface, and the helicoid of the second kind, respectively, in [2].

$$\begin{aligned}
&X_{eh}(u, v) = (-\sin u \cosh v, \cos u \cosh v, -u), \quad (u, v) \in \mathbb{R}^2,\\
&X_{ph}(a, b) = \left(-a + \frac{1}{3}a^3 - ab^2, -a^2 + b^2, -a - \frac{1}{3}a^3 + ab^2\right), \quad (a, b) \in \mathbb{R}^2,\\
&X_{hh}(\alpha, \beta) = (-\alpha, -\cosh\alpha \cos\beta, -\sinh\alpha \cos\beta), \quad (\alpha, \beta) \in \mathbb{R} \times S^1.
\end{aligned} \tag{2.6}$$

Being the conjugate of the hyperbolic catenoid, the helicoid of the second kind has the singularities at the lines $v = n\pi$. The surface is periodic with period 2π in the v direction; hence there are two singular curves in the (image of the) helicoid of the second kind whose velocity vectors form a hyperbola in the lightcone.

Note that $X_{*c}(x, -y) = -X_{*c}(x, y)$ and $X_{*h}(x, -y) = X_{*h}(x, y)$ where $* \in \{e, p, h\}$.

For any $\varphi \in (-\infty, 0]$, consider the following hyperbolic rotation of $\mathbb{L}^3$:

$$R_\varphi \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} \cosh\varphi & 0 & \sinh\varphi \\ 0 & 1 & 0 \\ \sinh\varphi & 0 & \cosh\varphi \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}. \tag{2.7}$$

The image of the timelike line $x = y = 0$ or a spacelike line $y = t = 0$ under R_φ is still a timelike or spacelike line, but its limit as $\varphi \to \pm\infty$ is a timelike line. This motivates that we may deform an elliptic or hyperbolic catenoid to a parabolic catenoid. It turns out that for any $(a, b) \in \mathbb{R}^2$

$$\frac{1}{2} e^{-2\varphi} R_\varphi \circ X_{\star c}(2e^\varphi a, 2e^\varphi b) \to X_{pc}(a, b) \quad \text{as } \varphi \to -\infty \tag{2.8}$$

where $\star \in \{e, h\}$. Later in Lemma 4.4, we apply the Björling formula to analyze this phenomena for arbitrary maximal surfaces with isolated singularities.

3. Singular Björling representation formula for maximal surfaces

Suppose that any $\gamma(u)$ of a smooth curve $\gamma : I \to \mathcal{U}$ is a shrinking singular point or a curvilinear singular point of a generalized maximal surface $(X, \mathcal{U})$. Here, the image of $X \circ \gamma$ may have corners as we see in Fig. 1. We first have the following

Lemma 3.1. *γ is a null curve.*

Proof. [1] $\langle \gamma', \gamma' \rangle = 0$ since $\langle , \rangle$ itself vanishes at singular points. $\square$

Therefore, in considering a singular Björling problem, we only consider null curves.

Now we want to prescribe one more datum. Recall that in the usual Björling formula, it appears as the unit normal vector field along the prescribed curve. The situation is somewhat different here since the unit normals of a generalized maximal surface on the singular curve are not well defined as vectors in $\mathbb{L}^3$.

But we observe that what we really need in the Björling problem is a curve and a family of tangent planes along the curve. So we consider a singular curve and a vector field which is perpendicular to the curve and whose length is the same as the speed of the curve. At this point we recall the well known fact that two null vectors are perpendicular to each other if and only if they are parallel to each other, and obtain the following formulation.

Singular Bjölring problem for generalized maximal surfaces. Given a real analytic null curve $\gamma : I \to \mathbb{L}^3$ and a real analytic null vector field $\mathcal{L} : I \to \mathbb{L}^3$ such that $\gamma'(u)$ and $\mathcal{L}(u)$ are proportional for all $u \in I$ and that at least one of $\gamma'(u)$ and $\mathcal{L}(u)$ is not identically $\vec{0}$, find a generalized maximal surface $X(u, v)$ with (u, v) as a conformal parameter whose u-parameter curve $X(u, 0)$ is $\gamma(u)$ and whose coordinate vector field $X_v(u, 0)$ along γ is $\mathcal{L}(u)$ for all $u \in I$.

We have the following answer to the above problem.

Theorem 3.2. *Given u, I, $\gamma = (\gamma_1, \gamma_2, \gamma_0)$ and $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_0)$ as above, define*

$$g(u) := \begin{cases} \left(\gamma_1'(u) + i\gamma_2'(u)\right)/\gamma_0'(u) & \text{if } \gamma' \not\equiv \vec{0}, \\ \left(\mathcal{L}_1(u) + i\mathcal{L}_2(u)\right)/\mathcal{L}_0(u) & \text{if } \gamma' \equiv \vec{0}. \end{cases} \tag{3.1}$$

If the analytic extension $g(z)$ of $g(u)$, where $z = u + iv \in \mathcal{U} \subset \mathbb{C}$, $I \subset \mathcal{U}$, satisfies

$$|g(z)| \not\equiv 1, \tag{3.2}$$

[1] We thank Umehara for giving us this short argument.

then there is exactly one generalized maximal surface $X : \mathcal{U} \to \mathbb{L}^3$ with $u + \mathrm{i}v$ as a conformal parameter and $X(u, 0) = \gamma(u)$, $X_v(u, 0) = \mathcal{L}(u)$. It is given by

$$X(u + \mathrm{i}v) = \gamma(u_0) + \mathrm{Re} \int_{u_0}^{z} \gamma'(w) - \mathrm{i}\mathcal{L}(w)\mathrm{d}w \tag{3.3}$$

where $u_0 \in I$ is fixed.

Proof. For the uniqueness, suppose there exists such a generalized maximal surface X. Then, $(X_u - \mathrm{i}X_v)(z)$ is holomorphic, and is equal to $\gamma'(u) - \mathrm{i}\mathcal{L}(u)$ on I. By analyticity, $(X_u - \mathrm{i}X_v)(z)$ must be the analytic extension of $\gamma'(u) - \mathrm{i}\mathcal{L}(u)$ which proves the uniqueness.

It is clear that the parametrization X given by (3.3) is a a generalized maximal surface which solves the singular Björling problem. Note that $g(z)$ in (3.2) is the Gauss map of X. □

Note that the conjugate maximal surface contains the integral curve of $\mathcal{L}$ as a singular curve and the normal direction at the curve is $-\gamma'$.

Corollary 3.3. *There is no generalized maximal surface which contains a piece of a null line as a singular curve.*

Proof. If there were such a surface, then we could reconstruct it by (3.3) with γ as a straight null line segment. However, for any null $\mathcal{L}$ with $\gamma' \perp \mathcal{L}$, the X in (3.3) is a straight null line segment, which is not a generalized maximal surface by definition. In fact, (3.2) is violated for any null $\mathcal{L}$ with $\gamma' \perp \mathcal{L}$ in this case. □

Suppose we are given a null curve $\delta(\theta)$ and a null vector field $\mathcal{N}(\theta)$, which are periodic in θ, such that $\dot{\delta}$ and $\mathcal{N}$ are perpendicular to each other. Then, because of the periodicity, we expect that the singular points are on the unit circle rather than on the real axis. This can be done as follows: Let $z = u + \mathrm{i}v$, $\xi = \mathrm{e}^{\mathrm{i}z} = r\mathrm{e}^{\mathrm{i}\theta}$. Then the u-axis in the z-plane is mapped to the unit circle in the ξ-plane. Now we observe

$$\phi = \begin{cases} 2X_z\mathrm{d}z = (X_u - \mathrm{i}X_v)\mathrm{d}z, \\ 2X_\xi\mathrm{d}\xi = \left(X_r - \dfrac{\mathrm{i}}{r}X_\theta\right)\mathrm{e}^{-\mathrm{i}\theta}\mathrm{d}\xi, \end{cases} \qquad \text{so } \phi = \begin{cases} 2X_z\mathrm{d}z = (X_u - \mathrm{i}X_v)\mathrm{d}u & \text{if } v \equiv 0, \\ 2X_\xi\mathrm{d}\xi = (X_\theta + \mathrm{i}X_r)\mathrm{d}\theta & \text{if } r \equiv 1. \end{cases}$$

Hence

$$X(r\mathrm{e}^{\mathrm{i}\theta}) = \delta(\theta_0) + \mathrm{Re} \int_{\mathrm{e}^{\mathrm{i}\theta_0}}^{\xi} \phi(\xi) \tag{3.4}$$

where $\phi(\xi)$ is the unique analytic extension of $(\dot{\delta}(\theta) + \mathrm{i}\mathcal{N}(\theta))\mathrm{d}\theta$. (By the analytic extension, we mean replacing $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{d}\theta$ by ξ and $-\mathrm{i}\mathrm{d}\xi/\xi$, respectively.)

So, a null curve and a null vector field do not determine a maximal surface in a unique way. The curve on the Riemann surface over which the two vector fields are parameterized must also be prescribed. Depending upon a choice of a parameter, the Björling data takes a different form.

Example 3.4. Consider

$$\delta(\theta) = \vec{0}, \qquad \mathcal{N}(\theta) = \frac{(\cos\theta, \sin\theta, 1)}{2 - 2\cos 2\theta}.$$

Then (3.4) yields $\phi = (\frac{1}{2}(\xi + \xi^{-1}), \frac{1}{2\mathrm{i}}(\xi - \xi^{-1}), 1)\frac{-\xi\mathrm{d}\xi}{(\xi^2-1)^2}$. If $\xi = \frac{\mathrm{e}^z+\mathrm{i}}{\mathrm{e}^z-\mathrm{i}}$ then $\phi = \mathrm{i}(\sinh z, 1, \cosh z)\mathrm{d}z$, which we can easily identify, from (2.5), as a null form for the hyperbolic catenoid. That is, these are Björling data for a hyperbolic catenoid.

Some of the interesting examples in [5] can be considered as generalizations of the hyperbolic catenoid through these Björling data. See Example 3.11.

We call (3.3) or (3.4) the *singular Björling representation formula* for maximal surfaces.

Convention: In the rest of this article, we use $\gamma(u)$, $\mathcal{L}(u)$ and $z = u + \mathrm{i}v$ to denote the Björling data if the singular set is parameterized by the real axis in the parameter domain of the maximal surface, but use $\delta(\theta)$, $\mathcal{N}(\theta)$ and $\xi = r\mathrm{e}^{\mathrm{i}\theta}$ if the singular set is parameterized by the unit circle in the parameter domain.

Table 1
Criteria for singularities in [9,17]

$\mathrm{Re}\left[\frac{g'}{g^2\hat{\omega}}\right] \neq 0$	$\mathrm{Im}\left[\frac{g'}{g^2\hat{\omega}}\right] \neq 0$	$\Leftrightarrow$ p is a cuspidal edge.	
$\mathrm{Re}\left[\frac{g'}{g^2\hat{\omega}}\right] \neq 0$	$\mathrm{Im}\left[\frac{g'}{g^2\hat{\omega}}\right] = 0$	$\mathrm{Re}\left[\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right] \neq 0$	$\Leftrightarrow$ p is a swallowtail.
$\mathrm{Re}\left[\frac{g'}{g^2\hat{\omega}}\right] \neq 0$	$\mathrm{Im}\left[\frac{g'}{g^2\hat{\omega}}\right] = 0$	$\mathrm{Re}\left[\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right] = 0$	$\Leftrightarrow$??
$\mathrm{Re}\left[\frac{g'}{g^2\hat{\omega}}\right] = 0$	$\mathrm{Im}\left[\frac{g'}{g^2\hat{\omega}}\right] \neq 0$	$\mathrm{Im}\left[\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right] \neq 0$	$\Leftrightarrow$ p is a cuspidal crosscap.
$\mathrm{Re}\left[\frac{g'}{g^2\hat{\omega}}\right] = 0$	$\mathrm{Im}\left[\frac{g'}{g^2\hat{\omega}}\right] \neq 0$	$\mathrm{Im}\left[\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right] = 0$	$\Leftrightarrow$??
$\mathrm{Re}\left[\frac{g'}{g^2\hat{\omega}}\right] = 0$	$\mathrm{Im}\left[\frac{g'}{g^2\hat{\omega}}\right] = 0$	$\Leftrightarrow$??	

Example 3.5 (*Reconstruction of the Elliptic, Parabolic, and Hyperbolic Catenoids*). Consider

$$\mathcal{N}_{ec}(u) = (\cos u, \sin u, 1), \qquad \mathcal{N}_{pc}(u) = (1-u^2, 2u, 1+u^2), \qquad \mathcal{N}_{hc}(u) = (1, \sinh u, \cosh u).$$

They are a circle, a parabola, and a hyperbola in the lightcone, which pass through $(1, 0, 1)$. The analytic extensions of $-\mathrm{i}\mathcal{N}_{ec}(u)\mathrm{d}u$, $-\mathrm{i}\mathcal{N}_{pc}(u)\mathrm{d}u$, $-\mathrm{i}\mathcal{N}_{hc}(u)\mathrm{d}u$ become the null form for the elliptic, parabolic, and hyperbolic catenoids, as we see from (2.5).

Since the conic sections form a continuous family, we may regard the various catenoids to be in a continuous family by this lemma.

Example 3.6 (*Perturbing the Helicoid of the First Kind*). Consider

$$\gamma(u) = (\sin u, -\cos u, u), \qquad \mathcal{L}(u) = \epsilon u(\cos u, \sin u, 1) \quad \text{where } \epsilon \in \mathbb{R}.$$

Note that $\epsilon u\gamma'(u) = \mathcal{L}(u)$. Then, using (3.3), the resulting maximal surface is

$$\begin{aligned} X(u, v) = {} & (\cosh v \sin u, -\cos u \cosh v, u) + \epsilon(-v \cosh v \sin u + \sinh v \sin u \\ & - u \cos u \sinh v, v \cos u \cosh v - \cos u \sinh v - u \sin u \sinh v, -uv). \end{aligned}$$

It has the Weierstrass data $g = \mathrm{e}^{\mathrm{i}z}$, $\mathrm{d}h = (1 - \mathrm{i}\epsilon z)\mathrm{d}z$. For any ϵ, the u-axis is the set of all singular points of X, whose image by X is the prescribed set $\gamma[(-\infty, \infty)]$. If $\epsilon = 0$, all the singularities are folded singularities, whose definition appears later in Definition 4.1. If $\epsilon \neq 0$, $z = 0$ is a cuspidal crosscap singularity and all other singularities are cuspidal edges, according to the criteria in Table 1.

Example 3.7 (*Perturbing the Elliptic Catenoid*). Fix $\epsilon \in \mathbb{R}$ and consider

$$\delta(\theta) = \epsilon\left(\cos\theta + \frac{1}{3}\cos 3\theta, -\sin\theta + \frac{1}{3}\sin 3\theta, \cos 2\theta\right), \qquad \mathcal{N}(\theta) = (\cos\theta, \sin\theta, 1).$$

Note that $\dot{\delta}(\theta) = -2\epsilon \sin 2\theta \mathcal{N}(\theta)$. Using (3.4), we obtain maximal surfaces with Weierstrass data $g = \xi$, $\mathrm{d}h = (1+\epsilon(\xi^2-\xi^{-2}))\xi^{-1}\mathrm{d}\xi$. When ϵ is small but nonzero, the resulting generalized maximal surface is a small perturbation around $\vec{0} \in \mathbb{L}^3$ of the elliptic catenoid, whose singular set is no longer a point but a curve in $\mathbb{L}^3$.

This example raises the following question: Is it possible to perturb conelike singularities of an arbitrary maximal surface so that they become another sort of singularities? This perturbation amounts to replacing isolated singularities by other sorts of singularities, and if successful, would be similar in spirit to the desingularization construction in [12], which replaces the intersections of coaxial catenoids in $\mathbb{E}^3$ by suitably modified pieces of Scherk surfaces. One interesting candidate for the substitute of a conelike singularity is the family of surfaces in [13].

It is possible to perturb an elliptic catenoid while maintaining the singular set as being still a conelike singularity. Any generalized maximal surface with a conelike singularity, other than that of the elliptic catenoid, may be thought to be a perturbation of the elliptic catenoid at least in a small neighborhood of the conelike singularity [5–8].

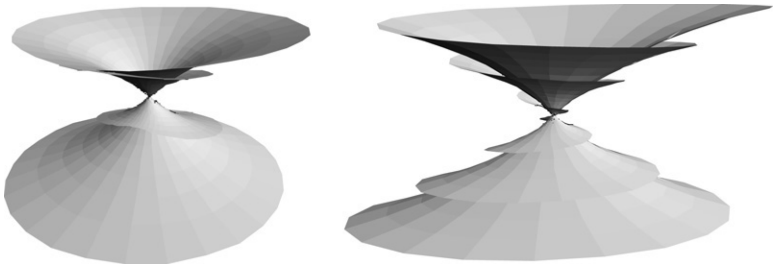

Fig. 2. The generalized maximal surfaces in Examples 3.8 and 3.9.

Now we construct interesting examples by choosing $\gamma(u), \mathcal{L}(u)$ or $\delta(\theta), \mathcal{N}(\theta)$ appropriately in (3.3) or (3.4), respectively.

Example 3.8 (*Immersed Isolated Singularity*). Consider

$$\delta(\theta) = \vec{0}, \qquad \mathcal{N}(\theta) := \left(1 + a\cos\frac{\theta}{2}\right)(\cos\theta, \sin\theta, 1), \quad a \in (-1, 1).$$

This produces an isolated singularity, the projection of the top or the bottom half of which into the xy-plane by $(x, y, t) \to (x, y)$ is double sheeted. See Fig. 2.

We may understand many properties of the above example by [6, Lemma 2.1] also.

Example 3.9 (*Helicoidal Shrunken Singularity*). Consider

$$\gamma(u) := \vec{0}, \qquad \mathcal{L}(u) := u(\cos u, \sin u, 1).$$

The image of $\mathcal{L}$ is a conical helix in the lightcone. This produces a shrunken singularity. See Fig. 2.

If the Weierstrass data g and $\mathrm{d}h$ of a generalized maximal surface are known, the Björling data for the generalized maximal surface can be derived as follows: At a singular point in $\mathcal{B}$, we have, using $g\bar{g} = 1$

$$X_u \Big/ \frac{1}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}z} + \overline{\frac{\mathrm{d}h}{\mathrm{d}z}}\right) = X_v \Big/ \frac{\mathrm{i}}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}z} - \overline{\frac{\mathrm{d}h}{\mathrm{d}z}}\right) = \left(\frac{1}{2}(\bar{g} + g), \frac{\mathrm{i}}{2}(\bar{g} - g), 1\right). \tag{3.5}$$

If the singularities are located on the u axis, then this formula is sufficient for deriving the Björling data. If the singularities are located on the unit circle $\mathrm{e}^{\mathrm{i}\theta}$ then

$$X_\theta = -r\sin\theta X_u + r\cos\theta X_v, \qquad X_r = \cos\theta X_u + \sin\theta X_v \quad \text{with } r = 1. \tag{3.6}$$

Example 3.10 (*[17, Example 5.2]*). If $g(\xi) = \xi$, $\mathrm{d}h = 2\xi\mathrm{d}\xi$, then

$$\delta(\theta) = \left(\cos\theta + \frac{1}{3}\cos 3\theta, -\sin\theta + \frac{1}{3}\sin 3\theta, \cos 2\theta\right), \qquad \mathcal{N}(\theta) = 2\cos 2\theta(\cos\theta, \sin\theta, 1),$$

up to a translation of $\mathbb{L}^3$. Note that $\dot{\delta}(\theta) = -\tan(2\theta)\mathcal{N}(\theta)$ and that the null curve δ has corners.

Example 3.11 (*[5]*). The generalized maximal surface in [5] with $g(\xi) = \xi$ and

$$\mathrm{d}h = \frac{-\mathrm{d}\xi}{(\xi - a)(\xi - a^{-1})}, \quad \text{or}$$

$$\mathrm{d}h = \frac{-\xi\mathrm{d}\xi}{(\xi^2 - a^2)(\xi^2 - a^{-2})}, \quad \text{or}$$

$$\mathrm{d}h = \frac{-\mathrm{d}\xi}{w(\xi - a^{-1})(\xi - b^{-1})} \quad \text{with } w^2 = \frac{(\xi - a)(\xi - b)}{(\xi - a^{-1})(\xi - b^{-1})}, \quad \text{or}$$

$$\mathrm{d}h = \frac{-\xi\mathrm{d}\xi}{w(\xi^2 - a^{-2})(\xi^2 - b^{-2})} \quad \text{with } w^2 = \frac{(\xi^2 - a^2)(\xi^2 - b^2)}{(\xi^2 - a^{-2})(\xi^2 - b^{-2})},$$

Table 2
Our criteria for singularities

$B_0 \neq 0$	$A_0 \neq 0$	$\Leftrightarrow z = 0$ is a cuspidal edge.	
$B_0 \neq 0$	$A_0 = 0$	$A_1 \neq 0$	$\Leftrightarrow z = 0$ is a swallowtail.
$B_0 \neq 0$	$A_0 = 0$	$A_1 = 0$	$\Leftrightarrow$??
$B_0 = 0$	$A_0 \neq 0$	$B_1 \neq 0$	$\Leftrightarrow z = 0$ is a cuspidal crosscap.
$B_0 = 0$	$A_0 \neq 0$	$B_1 = 0$	$\Leftrightarrow$??
$B_0 = 0$	$A_0 = 0$	$\Leftrightarrow$??	

for $0 < a < b < 1$ has, up to a translation of $\mathbb{L}^3$, the Björling data $\delta(\theta) = \vec{0}$ and

$$\mathcal{N}(\theta) = \frac{(\cos\theta, \sin\theta, 1)}{a - 2\cos\theta + a^{-1}}, \quad \text{or}$$
$$\mathcal{N}(\theta) = \frac{(\cos\theta, \sin\theta, 1)}{a^2 - 2\cos 2\theta + a^{-2}}, \quad \text{or}$$
$$\mathcal{N}(\theta) = \frac{(\cos\theta, \sin\theta, 1)}{\sqrt{a - 2\cos\theta + a^{-1}}\sqrt{b - 2\cos\theta + b^{-1}}}, \quad \text{or}$$
$$\mathcal{N}(\theta) = \frac{(\cos\theta, \sin\theta, 1)}{\sqrt{a^2 - 2\cos 2\theta + a^{-2}}\sqrt{b^2 - 2\cos 2\theta + b^{-2}}}.$$

It is the singly periodic generalized maximal surface with one conelike singularity, or the example in $L/\langle T\rangle$ with one singular point and nonparallel ends (i.e. the Scherk type maximal surfaces), or the example in $L/\langle T\rangle$ with two singular points and parallel ends, or the example in $L^3/\langle T_1, T_2\rangle$ with two singular points, respectively, in [5].

Now we proceed to pose a general method of constructing singularities of maximal surfaces, including the generic singularities of maxfaces which have been classified by Fujimori, Saji, Umehara, and Yamada [9,17]. First, we recall the criteria in Table 1 for cuspidal edge, swallowtail, cuspidal crosscap singularities of maxfaces in [9,17] where $\hat{\omega}$ is the analytic function such that $\mathrm{d}h = g\hat{\omega}\mathrm{d}z$ and all the functions are evaluated at p.

Now, consider

$$\phi(z) = (\phi_1(z), \phi_2(z), \phi_0(z)) = (\cos z, \sin z, 1)\,(\alpha(z) - \mathrm{i}\beta(z))\,\mathrm{d}z \tag{3.7}$$

where

$$\alpha(z) = A_0 + A_1 z + A_2 z^2 + \cdots, \qquad \beta(z) = B_0 + B_1 z + B_2 z^2 + \cdots,$$

for *real* numbers A_i, B_j. We assume that $\alpha(0) - \mathrm{i}\beta(0) \neq 0$, that is, $A_0 \neq 0$ or $B_0 \neq 0$. Then $g(z) = \phi_0(z)/(\phi_1(z) - \mathrm{i}\phi_2(z)) = \mathrm{e}^{\mathrm{i}z}$, $\hat{\omega}(z) = (\alpha(z) - \mathrm{i}\beta(z))\mathrm{e}^{-\mathrm{i}z}$; hence

$$\frac{g'}{g^2\hat{\omega}} = \frac{\mathrm{i}}{\alpha(z) - \mathrm{i}\beta(z)}, \qquad \frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)' = \frac{\alpha'(z) - \mathrm{i}\beta'(z)}{-(\alpha(z) - \mathrm{i}\beta(z))^2}.$$

Direct calculations show that

$$\frac{\mathrm{i}}{\alpha(0) - \mathrm{i}\beta(0)} = \frac{-B_0 + \mathrm{i}A_0}{A_0^2 + B_0^2},$$
$$\frac{\alpha'(0) - \mathrm{i}\beta'(0)}{-(\alpha(0) - \mathrm{i}\beta(0))^2} = \frac{[A_1(A_0^2 - B_0^2) + 2A_0B_0B_1] + \mathrm{i}[2A_0B_0A_1 - B_1(A_0^2 - B_0^2)]}{-(A_0^2 + B_0^2)^2}.$$

So the above criteria can be rewritten in terms of A_i's and B_j's as in Table 2.

So by choosing A_0 and B_0 appropriately, we can construct different characters of singularities of maximal surfaces. We believe that this construction will enable us to study the singularities which are not generic.

Remark 3.12. Consider $\phi(z) = \left(\cos(z^k), \sin(z^k), 1\right)(\alpha(z) - \mathrm{i}\beta(z))$ for some natural number $k \geq 2$ and $\alpha(z), \beta(z)$ as above. Then, $g(z) = \mathrm{e}^{\mathrm{i}z^k}$; hence the set of singular points is k number of lines intersecting at the origin. Note that in this case $\mathrm{Re}[\frac{g'}{g^2\hat{\omega}}(0)] = \mathrm{Im}[\frac{g'}{g^2\hat{\omega}}(0)] = 0$.

4. Shrunken singularities and folded singularities

We first set:

Definition 4.1. We call a curvilinear singular point $p \in \mathcal{U}$ a *folding* singular point if it has a neighborhood that can be reparameterized as $X : D_0(\epsilon) \subset \mathbb{C} \to \mathbb{L}^3$ with $p = 0$ such that the singular points are on the u-axis and $X_v(u, 0) \equiv 0$. $X(D_0(\epsilon))$ is called a folded singularity.

The name is motivated by the reflection principle stated in Theorem 4.3. We have the following relations between the isolated singularities and the folded singularities.

Lemma 4.2. *The conjugate maximal surface of a folded singularity is a shrunken singularity.*

Proof. Trivial from the various definitions. □

This explains the relationship between the shrinking singularities of the elliptic, parabolic, hyperbolic catenoids and the folded singularities of the helicoid of the first kind, Cayley's ruled surface, the helicoid of the second kind, respectively.

The following reflection principle is known for isolated singularities to Fernández, López, and Souam [8]. We state and prove it here for shrunken singularities and folded singularities.

Theorem 4.3. *Let $\mathcal{U} \subset \mathbb{C}$ be an open domain which contains an open real interval $I \subset \mathbb{R}$ and let $X, Y : \mathcal{U} \to \mathcal{L}$ be a generalized maximal surface with $z = u + \mathrm{i}v \in \mathcal{U}$ as a conformal parameter.*

(1) *If $X(u, 0) = (0, 0, 0)$ for any $u \in I$, then $X(u, -v) = -X(u, v)$ whenever they are defined.*
(2) *If $Y(u, 0)$ is a folded singularity for any $u \in I$, then $Y(u, v) = Y(u, -v)$ whenever they are defined.*

We call the above reflection principles with respect to a shrunken singularity and a folded singularity, respectively.

Proof. We may assume without loss of generality that Y is the conjugate surface of X. Let $\mathcal{L}(u) := X_v(u, 0)$ for $u \in I$ and let $\phi(z)$ be the analytic extension of $-\mathrm{i}\mathcal{L}(u)$. Then, $\psi(z) := -\overline{\phi(\bar{z})}$ is analytic since $\frac{\mathrm{d}}{\mathrm{d}\bar{z}}(-\overline{\phi(\bar{z})}) = -\overline{\frac{\mathrm{d}}{\mathrm{d}z}\phi(\bar{z})} = 0$, and coincides with ϕ on (a, b). Therefore, $\phi(z) = \psi(z) = -\overline{\phi(\bar{z})}$ everywhere. Then,

$$X(z) + \mathrm{i}Y(z) = \int_{u_0}^{z} \phi(z)\mathrm{d}z = -\int_{u_0}^{z} \overline{\phi(\bar{z})}\mathrm{d}z = -\overline{\int_{u_0}^{\bar{z}} \phi(\bar{z})\mathrm{d}\bar{z}} = -X(\bar{z}) + \mathrm{i}Y(\bar{z}).$$

Therefore, $X(z) = -X(\bar{z})$ and $Y(z) = Y(\bar{z})$, which prove the claims. □

An immediate consequence of the above reflection principle with respect to a shrunken singularity is that a generalized maximal surface with two shrunken singularities can be extended to a bigger one with an infinite number of shrunken singularities by successive reflections across the shrunken singularities.

Suppose X is a solution of the singular Björling problem with the null curve γ and the null vector field $\mathcal{L}$. Since

$$X(u, v) \approx X(u, 0) + X_v(u, 0)v = \gamma(u) + \mathcal{L}(u)v \quad \text{if } v \approx 0,$$

we see that γ and $\mathcal{L}$ determine the first-order shape of the singularities on γ, though this formula says nothing about the first-order shape of folded singularities. This explains the general shape of the shrunken singularities of the parabolic or the hyperbolic catenoid.

The same reasoning applied to a solution X of the singular Björling problem with the null curve δ and the null vector field $\mathcal{N}$ says that the first-order shape of X is given by

$$X(r\mathrm{e}^{\mathrm{i}\theta}) \approx X(\mathrm{e}^{\mathrm{i}\theta}) + X_r(\mathrm{e}^{\mathrm{i}\theta})(r - 1) = \delta(\theta) + \mathcal{N}(\theta)(r - 1) \qquad \text{if } r \approx 1.$$

In particular, if $\delta(\theta) \equiv 0$, then $X(r\mathrm{e}^{\mathrm{i}\theta}) \approx \mathcal{N}(\theta)(r - 1)$ for $r \approx 1$. Depending upon the behavior of $\mathcal{N}$, we get various kinds of isolated singularities. This provides a heuristic explanation of the behaviors of isolated singularities stated in [6, Lemma 2.1].

Fig. 3 shows isolated singularities with $\delta(\theta) = 0$ and $\mathcal{N}(\theta) = (\cos\theta, \sin\theta, 1)t(\theta)$ for $t(\theta) = 1 - \cos\theta, 1 + 2\cos\theta, 1 - \cos n\theta$ which are motivated by the cardioid, the limaçon, or the $2n$-leaved rose, respectively.

Fig. 3. Examples of isolated singularities.

As an application of the singular Björling representation formula, we show that the parabolic catenoid is the unique limit of the deformation of an arbitrary generalized maximal surface with a shrunken singularity by rotation and homothety, as follows: We may assume without loss of generality that around any shrinking singular point of a generalized maximal surface there is a local coordinate $z = u + \mathrm{i}v$ such that the Björling data for the surface are

$$\gamma(u) \equiv \vec{0}, \qquad \mathcal{L}(u) = (\cos f(u), \sin f(u), 1)\, t(u)$$

for some real analytic functions $f(u)$ and $t(u)$ with $f(0) = 0, t(0) = 1$.

Lemma 4.4. *Let R_φ be the hyperbolic rotation as in (2.7). Then $\mathrm{e}^\varphi R_\varphi : \mathbb{L}^3 \to \mathbb{L}^3$ deforms the above generalized maximal surface to the (n-sheeted) parabolic catenoid as $\varphi \to -\infty$.*

Proof. Let $f(u) = Au^n + \cdots$ for some $A \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$. Now we restrict our attention to the domain $\{z : |z| < C\mathrm{e}^{\varphi/n}\}$. Then it is easy to see that, if $\bar{u} = u\sqrt[n]{\frac{|A|}{2}\mathrm{e}^{-\varphi}}$, then

$$\mathrm{e}^{-\varphi} R_\varphi \circ \mathcal{L}(u) \to \left(1 - \bar{u}^{2n}, \frac{A}{|A|} 2\bar{u}^n, 1 + \bar{u}^{2n}\right) \quad \text{as } \varphi \to -\infty,$$

which is the Björling data for the (n-sheeted) parabolic catenoid. □

Note that the scaling factor in the lemma is different from that of (2.8).

5. Concluding remarks

At this moment, even though we can construct generalized maximal surfaces with prescribed singularities, we still do not have a full classification of singularities with $|g| = 1$ of generalized maximal surfaces. Furthermore, we do not know how the Björling data control the global behavior, such as embeddedness and periodicity, of the resulting generalized maximal surfaces. Their resolution will be particularly interesting for embedded periodic maximal surfaces with isolated singularities in the sense of [7].

The singular Björling formula will be useful in investigating the behavior of ends with singularities of generalized maximal surfaces. For example, if $\alpha(z)$ in (3.7) has a pole at 0, then the resulting generalized maximal surface has an end with singularities.

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